Reverse flow and supersonic interference

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First, from a volumetric formulation of the momentum theorem of linearized theory, a general analytic proof is presented of the invariance of the drag of an arbitrary spatial distribution of horseshoe vortices and sources under reversal of the undisturbed flow. By consideration of the interference drag of two such singularity distributions, a reverse-flow relation for steady subsonic or supersonic flow is then obtained. This relation, a generalization of the Ursell–Ward theorem, may be applied to configurations with bodies whose surfaces are not quasi-cylindrical and whose surface pressures are quadratically related to the perturbation velocity.

The relation is used to discuss several interfering two-body arrangements in supersonic flow. It is shown that, in certain cases, the drag and lift may be determined without knowledge of the interference flow field associated with the arbitrarily prescribed body geometry. The simplicity of the results permits the formulation of optimum problems. The invariance of the drag under flow reversal with unchanged geometry is also established.

Introduction

The development of the reverse-flow or reciprocity relations of linear small-disturbance theory was initiated by the discovery of the invariance of the pressure drag of symmetric non-lifting wings in supersonic flow with respect to reversal of the flow direction (von Kármán 1947). The subject was treated independently on a considerably more general basis at about the same time by Hayes (1947). The theoretical and practical ramifications of subsequent contributions are extensive and need not be recounted here. This research, however, has resulted in two principal theorems which are of interest in the present paper.

In connexion with the first, there is considered lineal, planar, or spatial distributions of horseshoe vortices and sources, singularities sufficient† to produce the same effect in the stream as any body or arrangement of bodies.‡ It is shown

† When the vorticity appearing in the general solution for the perturbation field is distributed over a cylindrical reference surface or a volume, it may be re-expressed as horseshoe vorticity.

‡ In this paper, the word body refers to any obstacle to which perturbation flow theory is applicable. By fusiform body is meant a body of fuselage or projectile shape. A slender body is a fusiform body whose section by meridional or transverse planes exhibits the degree of regularity required in the theory of Ward (1949). A quasi-cylindrical body is one whose boundary conditions and surface pressures may, within the approximation, be discussed on a cylindrical reference surface, with generators parallel to the undisturbed stream, which is near the actual surface (e.g. a pair of wing panels supported by an infinite body whose shape differs but slightly from a circular cylinder with finite diameter).

that the drag of any distribution of these two singularities is unchanged if, in the undisturbed stream, the sign of the velocity vector is changed without change in Mach number or density (Hayes 1947). The sign of the singularity strengths may be reversed with the flow. Hayes' theorem does not deal explicitly with the geometry of the system represented which is, in general, different in the reverse flow. The theorem has been discussed for planar wings in connexion with the development of conditions for minimum drag in the combined flow field (Jones 1951) and for more general arrangements of bodies in the same connexion (Jones 1956). The proofs are based on physical arguments concerning the interference between two discrete singularities in forward and reverse flow, and involve the superposition of the pressure and cross-flow induced by each in a reference plane containing the two singularities; the arguments do not deal explicitly with cases where the bodies are not quasi-cylindrical or the pressure is not linearly related to the perturbation velocity.

The second theorem mentioned is the Ursell–Ward theorem, which contains virtually all of the other theorems developed thus far for steady flow as special cases (Ursell & Ward 1950; Heaslet & Spreiter 1953). This is an identity involving surface integrals related to the drag integral, and connecting a forward flow past a quasi-cylindrical body (or bodies) to the reverse flow past another such body which has the same reference surface; the linear pressure relationship pertains. The theorem can be used to relate the flow past a body with specified geometry or pressure distribution to the reverse flow past a body whose geometry or pressure distribution may be independently specified. The two bodies may therefore be identical. As is evident in the formulation used in the present paper, this theorem contains the first theorem as a special case, but only for the quasi-cylindrical bodies to which it may be applied.

Starting from a volumetric formulation of the momentum theorem, this paper presents, as a preliminary, a general analytic proof of Hayes' theorem which clarifies aspects of the physical derivation discussed above. There is then obtained a reverse-flow theorem for steady subsonic or supersonic flow which is related to the Ursell–Ward theorem, but which may be applied as well to arrangements containing bodies whose surfaces are not quasi-cylindrical and whose surface pressures are not linearly related to the perturbation velocity. Whereas such bodies need not be identical in forward and reverse flow, they must be related in a manner to be discussed; this geometric restriction is concomitant with the relaxation of the two aforesaid restrictions. The relation is a generalization, in the sense just indicated, of the Ursell–Ward theorem, and it is possible to obtain the surface representation of the results presented with a similar derivation. However, the relation is more generally formulated with use of the equality, in forward and reverse flow, of the interference drags of two spatial singularity distributions, the first associated with the bodies in forward flow and the second with the bodies in reverse flow. The underlying unity of the two theorems just discussed is therefore established, and a view consistent with the extensive interference literature is maintained. The present relation contains, in fact, the two theorems as special cases.

In the second portion of the paper, the reverse-flow relation is used to discuss the drag and lift of several interfering two-body arrangements in supersonic flow. The arrangements considered not only serve to exhibit some implications of the theorem, but are of theoretical and practical interest in themselves. The applicability of the relation is, however, by no means restricted to these specific configurations.

Considerable attention has recently been devoted to the problem of achieving favourable interference in arrangements of bodies in a moderate supersonic flow, and in some cases optimum problems have been posed in very general terms. Reference is made to the methods of Jones (1956), Lomax (1955), Lomax & Heaslet (1956*a*) and E. W. Graham, Lagerstrom, Licher & Beane (1957). Frequently, problems are set such that the orientations and singularity distributions of bodies in the arrangement are sought which reduce or minimize the drag of the system, subject to restraints on lift, volume, etc. (M. E. Graham 1955; Licher 1955; Lomax & Heaslet 1956*b*; Friedman & Cohen 1954). When the bodies interfere, each induces a cross-flow on the other surfaces, resulting in a distortion of the isolated boundaries associated with the singularities of each. The geometry of the arrangement is usually discussed only in general terms.

When an optimum, or otherwise suitable, geometry is arrived at by such methods, it is necessary in practice to determine the flow field, and particularly the drag and lift, produced by these boundaries for other values of incidence and Mach number. In any case, when the geometry of an arrangement is prescribed, cancellation of the mutual cross-flow by introduction of an interference flow is required. As illustrated by the wing-fuselage interference problem (see Lawrence & Flax 1954), the labour in determining the interference flow defined by this classical indirect formulation is great. Depending on how the problem is posed, favourable shapes might remain to be sought subsequent to the calculation.

In the present paper, the reverse-flow relation is first used to establish the invariance of the drag of the two-body arrangements under flow reversal with unchanged geometry. It is then shown that it is possible, for supersonic flow, to determine the drag and lift of certain of these interfering body arrangements, whose geometry is arbitrarily prescribed, with no knowledge of the interference flow field. The forces are constructed from the solution for the flow field of each body when it appears isolated in forward flow and in reverse flow. The determination is possible when the interference flow produced on each body does not influence the other; when this restriction is not satisfied, an incomplete knowledge of the interference flow suffices for the determination of the aerodynamic forces. The method uses the reverse flow theorem to relate the force appearing when the two bodies interfere without mutual distortion in forward flow to the force appearing when the same bodies interfere with mutual distortion in the hypothetical reverse flow. The relations obtained are of sufficient simplicity to permit formulation of optimum problems. The method lends itself to application to other related arrangements.

The aerodynamic force

Consider the steady, homentropic flow of a compressible gas past an arrangement of bodies whose inclinations are small with respect to the remote uniform stream with velocity

$$\mathbf{U} = U\mathbf{i}, \tag{1}$$

where $U > 0$ in forward flow and $U < 0$ in reverse flow, $\mathbf{i}$ is a unit vector in the positive direction of the cartesian co-ordinate x, and $\mathbf{j}$ and $\mathbf{k}$ are similarly defined for the cartesian co-ordinates y and z. If the fluid velocity $\mathbf{q}$ is written

$$\mathbf{q} = \mathbf{U}+\mathbf{V}, \tag{2}$$

then the perturbation velocity

$$\mathbf{V} = u\mathbf{i}+v\mathbf{j}+w\mathbf{k} \tag{3}$$

is governed by the equation

$$\nabla.\mathbf{W} = 0 \tag{4}$$

together with the definitions

$$\mathbf{W} = \mathbf{\Phi}.\mathbf{V}, \quad \mathbf{\Phi} = \begin{bmatrix} -B^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5}$$

where $B^2 = M^2-1$, M being the Mach number of the undisturbed stream. Further,

$$\nabla\times\mathbf{V} = 0 \tag{6}$$

exterior to any regions of trailing vorticity.

The aerodynamic force on the bodies may be evaluated through use of the momentum theorem. If equation (2) and the continuity equation are used therein, and the pressure p and density ρ are then eliminated by means of the approximate relations

$$(p-p_0)/\rho_0 = -\mathbf{U}.\mathbf{V}-\tfrac{1}{2}\mathbf{V}.\mathbf{W} \tag{7}$$

and

$$\rho/\rho_0 = 1-(M^2/U^2)\,\mathbf{U}.\mathbf{V}, \tag{8}$$

where the subscript 0 denotes conditions in the undisturbed stream, the momentum theorem appears in the form (Ward 1955*a*)

$$\mathbf{F} = \rho_0\mathbf{U}\times\oint_S \mathbf{n}\times\mathbf{V}\,dS+\rho_0\oint_S(\tfrac{1}{2}\mathbf{V}.\mathbf{W}\mathbf{n}-\mathbf{V}\mathbf{W}.\mathbf{n})\,dS. \tag{9}$$

Here S is any surface with outward unit normal $\mathbf{n}$ which completely encloses the bodies, and $\mathbf{F}$ is the total aerodynamic force on the bodies, provided that the total source strength enclosed by S is zero. The first term evidently gives a force perpendicular to the undisturbed stream.

Equation (9) is correct within the quasi-cylinder approximation

$$t \ll 1, \tag{10}$$

t being the appropriate thickness ratio, or the slender body approximation

$$t^2\log t \ll 1, \tag{11}$$

but, depending on the choice of S, may contain terms of higher order than required. A lower approximation to (7) and (8), and therefore to (9), is often used for slender bodies which is, in fact, slightly too low if the bodies are closed. It is assumed that (9) is also adequate for arbitrary fusiform bodies, the approximation being (Lighthill 1945, 1954)

$$u \ll 1, \tag{12}$$

and for arbitrary body arrangements. The condition (12) is about equivalent to (11) but may fall at least to (10) locally, as at shoulders. Within approximations (10), (11) or (12), the boundary condition on the surfaces A of the bodies

$$(\mathbf{U}+\mathbf{V}).\mathbf{n} = 0 \tag{13}$$

may be written

$$(\mathbf{U}+\mathbf{W}).\mathbf{n} = 0, \tag{14}$$

essentially because $\mathbf{V}$ and $\mathbf{W}$ differ only in their x-components and the x-component of $\mathbf{n}$ is small. The momentum flux given by (9) vanishes within the approximation of (14) to (13) when S is chosen as A. If the drag D is defined as the component of $\mathbf{F}$ in the direction of flow, if edge forces are neglected and S is taken as A, and if use is made of (14), then (9) gives for the drag

$$D = \frac{U}{|U|}\mathbf{F}.\mathbf{i} = \frac{U}{|U|}\rho_0 \oint_A (\mathbf{U}.\mathbf{V}+\tfrac{1}{2}\mathbf{V}.\mathbf{W})\,\mathbf{n}.\mathbf{i}\,dS, \tag{15}$$

which is consistent with a surface integration of the pressure given by (7). For quasi-cylinders, however, the approximation to the pressure

$$(p-p_0)/\rho_0 = -\mathbf{U}.\mathbf{V} = -Uu \tag{16}$$

suffices, whereas for slender bodies the relation (Ward 1949)

$$(p-p_0)/\rho_0 = -Uu-\tfrac{1}{2}(v^2+w^2) \tag{17}$$

is required on or near the surface. Equation (17) is still correct for arbitrary fusiform bodies (Lighthill 1945, 1954). The term $-\frac{1}{2}B^2u^2$ in (15) may then always be discarded.

Let the bodies be represented by a spatial distribution of vorticity and sources over the regions T interior to the surfaces A of the bodies (a thickness distribution can temporarily be ascribed to a plate wing). The spatial distribution corresponding to a given body is not generally unique, and this freedom might be used to smooth out any singularities appearing in the intensities of surface or lineal distributions. With the body interiors regarded as part of the fluid domain, the surface integrals in (9) may be transformed to volume integrals. The identity appropriate to the second integral follows from (34) below by discarding subscripts. Equation (9) may then be written (Ward 1955*b*)

$$\mathbf{F} = \rho_0 \mathbf{U} \times \int_T \nabla \times \mathbf{V}\, dT + \rho_0 \int_T [\mathbf{W} \times (\nabla \times \mathbf{V}) - \mathbf{V}\nabla.\mathbf{W}]\, dT. \tag{18}$$

Integration over the volume of each body gives the force on that body. A more direct derivation of (18) leads to difficulties in obtaining the correct approximation to the integrands for slender and other fusiform bodies.

For a spatial distribution of vorticity with intensity $\boldsymbol{\Omega}$ and sources with an intensity f such that the mass flux per unit volume is $\rho_0 f$, the linearized equations for $\mathbf{V}$ which replace (4) and (6) are (Ward 1955*a*)

$$\nabla.\mathbf{W} = f, \tag{19}$$

and

$$\nabla \times \mathbf{V} = \boldsymbol{\Omega}, \tag{20}$$

where f and $\mathbf{\Omega}$ are related to the geometry. The conservation in space of vorticity is assured by the relation

$$\nabla . \mathbf{\Omega} = 0, \tag{21}$$

which follows from (20). Let the vorticity at each point be resolved according to the expression

$$\mathbf{\Omega} = \Omega_x \mathbf{i} + \boldsymbol{\omega}, \tag{22}$$

where $\boldsymbol{\omega}$ is a vector perpendicular to the x-axis which is referred to here as bound vorticity. If the drag is formed from (18), and if (5), (19), (20) and (22) are used, there results after slight rearrangement

$$D = -\rho_0 \frac{U}{|U|} \int_T (\mathbf{i} \times \boldsymbol{\omega} . \mathbf{V} + f \mathbf{i} . \mathbf{V})\, dT. \tag{23}$$

Comparison with the lateral force in (18), a version of the Kutta–Joukowski relation, shows that the term $\rho_0 \mathbf{U} \times \boldsymbol{\omega}$ is the elemental lateral force perpendicular to the bound vorticity at each point; comparison with (16) shows that $-\rho_0 \mathbf{U} . \mathbf{V}$ is equal to the linear homentropic pressure difference. The first term is then the product of the elemental lateral force and the component of the cross-flow in its direction. The general volumetric representation of drag then yields a result closely resembling, in its two terms, the familiar relations of wing theory for a planar distribution of lift and thickness, respectively. Its significance stems from its applicability to the more restrictive fusiform bodies as well as to quasi-cylinders, so that no distinction between the classes of bodies need be made over T. Thus (23) permits, as for quasi-cylinders, the convenient decomposition of the drag of a distribution of singularities into self-induced drag and various interference drags (E. W. Graham *et al.* 1957). On the other hand, the formulation (15) requires, for bodies which are not quasi-cylindrical, integration of the product of the quadratic pressure and cross-flow over the actual body surfaces produced by the collective effect of all singularities; the drag decomposition mentioned is not possible. Equation (9), with a truncated cylindrical surface S, for example, provides a reference surface and quadratic products permitting drag decomposition for both classes of bodies, but in this formulation each point on the body surfaces A corresponds to a region of S, and vice versa.

The exhibited properties of the momentum theorem have their counterparts in the reverse flow relations to follow.

The perturbation velocity field

A solution to (19) and (20) has been given by Ward (1952*a*,*b*, 1955*a*), the appropriate form of which is

$$\mathbf{V}(\mathbf{r}) = -\frac{1}{2\pi\epsilon} {}^{*}\!\!\int_d f(\boldsymbol{\rho}) \nabla \frac{1}{R} d\tau - \frac{B^2}{2\pi\epsilon} {}^{*}\!\!\int_d \mathbf{\Omega}(\boldsymbol{\rho}) \times (\mathbf{r} - \boldsymbol{\rho}) \frac{1}{R^3} d\tau, \tag{24}$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\boldsymbol{\rho} = \xi\mathbf{i} + \eta\mathbf{j} + \zeta\mathbf{k}$, $R = \sqrt{\{(x-\xi)^2 - B^2[(y-\eta)^2 + (z-\zeta)^2]\}}$, $\epsilon = 1$ for $M > 1$ and $\epsilon = 2$ for $M < 1$, * denotes Hadamard's finite part of an integral, and d denotes the domain of dependence of the point $\mathbf{r}$: the region of the space τ interior to the fore Mach cone from $\mathbf{r}$ in supersonic flow; the entire space

τ in subsonic flow. The second integral is recognized as a compressible generalization of the Biot–Savart law. It may be recast as horseshoe vorticity by constructing elemental trailing vortex pairs and articulating these to the bound vorticity at each point. Then (24) becomes

$$\mathbf{V}(\mathbf{r}) = \frac{-1}{2\pi\epsilon}{}^{*}\int_d \left\{f(\boldsymbol{\rho})\nabla\frac{1}{R} + B^2\boldsymbol{\omega}(\boldsymbol{\rho})\times(\mathbf{r}-\boldsymbol{\rho})\frac{1}{R^3} + [\boldsymbol{\omega}(\boldsymbol{\rho}).\nabla][\mathbf{i}\times(\mathbf{r}-\boldsymbol{\rho})\psi(\mathbf{r},\boldsymbol{\rho})]\right\}d\tau, \tag{25}$$

where
$$\psi(\mathbf{r},\boldsymbol{\rho}) = \frac{1}{[(y-\eta)^2+(z-\zeta)^2]}\left[\frac{U}{|U|}(\epsilon-1)+\frac{(x-\xi)}{R}\right]. \tag{26}$$

The first term in (25) gives the effect of the sources, the second the effect of the bound vorticity, and the third the effect of the trailing vortex pairs.

Equations (24) and (25) are invariant under a reflexion of the x- and y-axes, and therefore hold in reverse as well as forward flow.

Invariance of drag of singularity distribution under flow reversal

It may now be shown that the drag of the distribution of horseshoe vortices and sources over the regions T is unchanged by a reversal of the direction of the undisturbed stream.† Let $U = U_F > 0$, the subscript F referring to forward flow. Substitution of (25) into (23) gives for the drag in forward flow

$$D_F = -\rho_0\int_T dT\,{}^{*}\int_d d\tau[\mathbf{i}\times\boldsymbol{\omega}(\mathbf{r})+f(\mathbf{r})\,\mathbf{i}].\left(\frac{-1}{2\pi\epsilon}\right)\left\{f(\boldsymbol{\rho})\nabla\frac{1}{R}\right.$$
$$\left.+\frac{B^2}{R^3}\boldsymbol{\omega}(\boldsymbol{\rho})\times(\mathbf{r}-\boldsymbol{\rho})+[\boldsymbol{\omega}(\boldsymbol{\rho}).\nabla][\mathbf{i}\times(\mathbf{r}-\boldsymbol{\rho})\psi_F(\mathbf{r},\boldsymbol{\rho})]\right\}. \tag{27}$$

It is desired to reverse the order of the two volume integrations. Such an operation poses no special problems for subsonic flow, but for supersonic flow the singular integrands can pose difficulties in differentiating or integrating under an integral sign; the difference in approach to this problem distinguishes, in part, the research of Volterra and Hadamard concerning the wave equation. Reference is made to the definitions and approach of Lomax, Heaslet & Fuller (1951) or Heaslet & Lomax (1954). The operation of integration under the finite part sign can in fact be performed, and without the appearance of accretive terms, provided that the two volume integrals are properly iterated. If the co-ordinate net is such that the inner or first integration in each case proceeds along a subsonic curve,‡ and the two remaining integrations proceed along supersonic curves, then the proposed reversal is valid. The limit of the inner integration with respect to T becomes, from the geometry, the domain of influence δ of the point $\boldsymbol{\rho}$. The outer integration with respect to τ extends over all of the regions T. Physically, the drag of the horseshoe vortex and source at $\boldsymbol{\rho}$ is now being computed before summing over like singularities. The finite-part concept and the present preferred order of iteration used in both (23) and (25) together allow the singularities to

† It is understood that M and ρ_0 are unchanged under reversal.

‡ A subsonic curve has the property that each successive point, when the curve is traversed in one of the two possible senses, lies in the domain of influence of the preceding point. When no two points on a curve have this property, the curve is supersonic.

be treated as entities, each possessing a flow field and inducing forces like discrete singularities in incompressible flow. This freedom is implicit in the physical arguments of Hayes and Jones.

By formal expansion of the products and rearrangement, the integrand of (27) can be written

$$-[\mathbf{i}\times\boldsymbol{\omega}(\boldsymbol{\rho})+f(\boldsymbol{\rho})\,\mathbf{i}]\,.\left(\frac{-1}{2\pi\epsilon}\right)\left\{f(\mathbf{r})\,\nabla_G\frac{1}{R}\right.$$
$$\left.+\frac{B^2}{R^3}\boldsymbol{\omega}(\mathbf{r})\times(\boldsymbol{\rho}-\mathbf{r})+[\boldsymbol{\omega}(\mathbf{r})\,.\nabla_G]\,[\mathbf{i}\times(\boldsymbol{\rho}-\mathbf{r})\,\psi_R(\boldsymbol{\rho},\mathbf{r})]\right\}, \quad (28)$$

where $$\nabla_G\equiv\mathbf{i}\frac{\partial}{\partial\xi}+\mathbf{j}\frac{\partial}{\partial\eta}+\mathbf{k}\frac{\partial}{\partial\zeta},$$

and $$\psi_R(\boldsymbol{\rho},\mathbf{r})=-\psi_F(\mathbf{r},\boldsymbol{\rho})$$

from (26), the subscript R referring to the reverse flow $U_R=-U_F<0$. The first term is recognized from (23) as the expression for the elemental drag in reverse flow; the second is recognized from (25) as the expression for the elemental velocity in reverse flow with the roles of $\mathbf{r}$ and $\boldsymbol{\rho}$ interchanged. This result therefore means that the drag induced on the horseshoe vortex and source at $\mathbf{r}$ by the horseshoe vortex and source at $\boldsymbol{\rho}$ in forward flow, is equal to the drag induced on the horseshoe vortex and source at $\boldsymbol{\rho}$ by the horseshoe vortex and source at $\mathbf{r}$ in reverse flow. Trailing vortex pairs induce no streamwise perturbations and hence do not interfere with sources.

Upon substitution of (28) into (27), written with the reversed order of integration, and comparison with (25), there results

$$D_F=\rho_0\int_T d\tau[\mathbf{i}\times\boldsymbol{\omega}(\boldsymbol{\rho})+f(\boldsymbol{\rho})\,\mathbf{i}]\,.\,\mathbf{V}_R(\boldsymbol{\rho}), \quad (29)$$

since δ becomes the domain of dependence of the point $\boldsymbol{\rho}$ in the reverse flow U_R. As was just noted from (23), the right member of (29) is the drag of the distribution in reverse flow. Thus

$$D_F=D_R \quad (30)$$

for subsonic or supersonic flow. (The contribution to this result of any wing edge forces can be considered separately with the approach of Ursell & Ward (1950). It is possible to demonstrate that the statement (30) includes any edge forces and therefore holds for the complete pressure drag.)

It is customary to reverse with the flow the signs of $\boldsymbol{\omega}$ (or $\boldsymbol{\Omega}$) and f in order to maintain under reversal the same lateral force (18) and real volumes. This does not change the above conclusions.

The shape of the bodies will, in general, be different in the reverse flow so that A, and therefore T, should be different. This distortion under reversal is not given properly here because the singularities have been attached to the space instead of associated with a fictitious captive fluid simulating the bodies and distorting with them. The effect is evidently of no concern in quasi-cylinder theory, and provided that the boundary conditions and pressure are also discussed on the undistorted surfaces instead of the true surfaces, the effect is also negligible in linear theory for more general bodies. The singularities need not, of course, occupy the entire region interior to each body.

A reverse-flow relation

Consider an arrangement of bodies represented in the forward subsonic or supersonic flow U_F by a distribution of horseshoe vortices $\boldsymbol{\omega}_F$ and sources f_F over the connected or unconnected regions T interior to the bodies, and such that the induced flow field satisfies

$$\mathbf{W}_F . \mathbf{n} = -\mathbf{U}_F . \mathbf{n} \tag{31}$$

on the body surfaces A with outward normal $\mathbf{n}$. Also consider a related arrangement of bodies represented in the reverse flow U_R by a different distribution $\boldsymbol{\omega}_R$ and f_R over the same regions T. Suppose that both distributions now appear together in forward flow and also in reverse flow. Let $D_{F,RF}$ be the interference drag in forward flow induced on the latter distribution by the former; let $D_{R,FR}$ be the interference drag in reverse flow induced on the former distribution by the latter. A slight modification of the argument of the last section gives a property of two distinct, interfering distributions under flow reversal. There appears at once the desired reverse-flow relation connecting the original forward and reverse flows:

$$\begin{aligned} D_{F,RF} &\equiv -\rho_0 \int_T (\mathbf{i} \times \boldsymbol{\omega}_R . \mathbf{V}_F + f_R \mathbf{i} . \mathbf{V}_F) \, dT \\ &= \rho_0 \int_T (\mathbf{i} \times \boldsymbol{\omega}_F . \mathbf{V}_R + f_F \mathbf{i} . \mathbf{V}_R) \, dT \equiv D_{R,FR}. \end{aligned} \tag{32}$$

The preceding relation (30) evidently applies to the special case of (32) where the distributions in forward and reverse flow are the same. With use of (5), (19), (20) and (22), the above may be written

$$\mathbf{U}_F . \int_T [\mathbf{V}_F \nabla . \mathbf{W}_R + \mathbf{V}_R \nabla . \mathbf{W}_F - \mathbf{W}_F \times (\nabla \times \mathbf{V}_R) - \mathbf{W}_R \times (\nabla \times \mathbf{V}_F)] \, dT = 0. \tag{33}$$

In order to discuss boundary conditions and surface pressures, the integration over the regions interior to any or all of the bodies may be transformed to an integral over the corresponding surfaces of A by means of the volume-surface integral identity

$$\begin{aligned} &\int_{T'} [\mathbf{V}_F \nabla . \mathbf{W}_R + \mathbf{V}_R \nabla . \mathbf{W}_F - \mathbf{W}_F \times (\nabla \times \mathbf{V}_R) - \mathbf{W}_R \times (\nabla \times \mathbf{V}_F)] \, dT \\ &\qquad = \oint_{A'} (\mathbf{V}_F \mathbf{W}_R . \mathbf{n} + \mathbf{V}_R \mathbf{W}_F . \mathbf{n} - \mathbf{V}_F . \mathbf{W}_R \mathbf{n}) \, dS, \end{aligned} \tag{34}$$

where $\mathbf{n}$ is the outward unit normal to A'. This identity was used by Ursell & Ward (1950; see also Ward 1955*a*), and, when the subscripts simply identify two solutions, it serves for the construction of solutions to the simultaneous equations (19) and (20) as does Green's theorem for the related formulation of the theory in terms of the potential.

Unlike the corresponding transformation for the aerodynamic force, the use of (34) in (33) does not immediately give integrals of the form of (15) except for quasi-cylinders, in which case the last term in (34) is negligible and the Ursell–Ward theorem results at once. The desired mixed pressure-slope products can be constructed by adding and subtracting terms, however, and the accretive terms will be shown in the next section to vanish; use of (31) and the relations

$$\mathbf{U}_R = -\mathbf{U}_F, \quad \mathbf{V}_F.\mathbf{W}_R = \mathbf{W}_F.\mathbf{V}_R$$

then gives the identity

$$\mathbf{U}_F.\oint_{A'}(\mathbf{V}_F\mathbf{W}_R.\mathbf{n}+\mathbf{V}_R\mathbf{W}_F.\mathbf{n}-\mathbf{V}_F.\mathbf{W}_R\mathbf{n})\,dS = \oint_{A'}[(\mathbf{U}_F.\mathbf{V}_F+\tfrac{1}{2}\mathbf{V}_F.\mathbf{W}_F)\,\mathbf{W}_R.\mathbf{n} - (\mathbf{U}_R.\mathbf{V}_R+\tfrac{1}{2}\mathbf{V}_R.\mathbf{W}_R)\,\mathbf{W}_F.\mathbf{n}-\tfrac{1}{2}(\mathbf{V}_F+\mathbf{V}_R).(\mathbf{W}_F\mathbf{W}_R.\mathbf{n}-\mathbf{W}_R\mathbf{W}_F.\mathbf{n}) + \tfrac{1}{2}\mathbf{W}_F.\mathbf{V}_R(\mathbf{W}_F+\mathbf{W}_R).\mathbf{n}]\,dS. \quad (35)$$

When a body is not a quasi-cylinder, the choice of the surface of the related body in reverse flow is, by definition, limited.

The reverse-flow relation (32) or (33), together with the adjoining identities (34) and (35), can be used to provide information concerning the aerodynamic force on any body or arrangement of bodies to which linearized theory is applicable. Formulation in terms of volume integrals as well as surface integrals proves useful. The required relationships have been formally assembled here; their nature and implications are exhibited in the applications below.

Two-body arrangements in supersonic flow

Discussion

In this section the reverse-flow relation is used in several different ways for the determination of the drag and lift in supersonic flow of the interfering two-body arrangements shown in figure 1. Illustrated in figure 1 (*a*) is a fusiform body

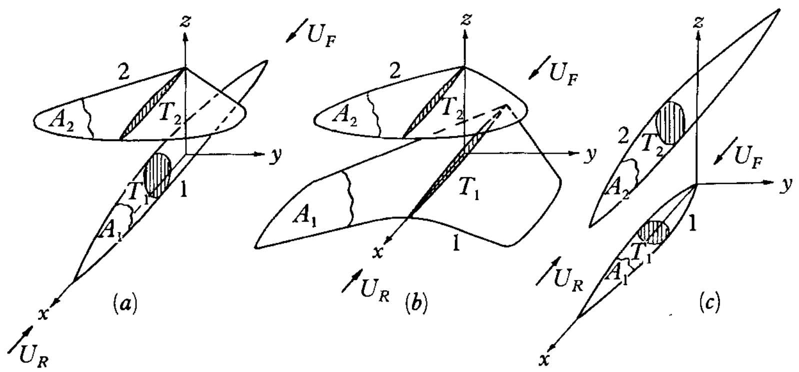

FIGURE 1. Two-body arrangements.

positioned under a wing; figure 1 (*b*) depicts a biplane; figure 1 (*c*) illustrates two fusiform bodies. The perturbation flow field in each case is generated by singularities distributed over the regions T_1 and T_2 interior to the bodies 1 and 2 with areas A_1 and A_2, respectively. The flow field of each isolated body, respectively $\mathbf{V}_1$ and $\mathbf{V}_2$, is assumed known in both forward and reverse flow; the analytical methods required are well known and need not be discussed here.

Most features of the three arrangements appear in a discussion of the first, which exhibits the mixed interfering boundary types: the fusiform body is not a quasi-cylinder whereas the wing is a plane quasi-cylinder. Linearized theory gives the isolated flow field of the former to the approximation (11) and the latter to the approximation (10). Since the interference does not modify these considerations, the relative magnitudes of the two thickness ratios are taken to be related by

$$t_2 = O[t_1^2 \log t_1], \tag{36}$$

which defines the relative magnitudes of the perturbations. The order of approximation for the arrangement remains as in (11) in terms of t_1. The perturbation velocity satisfies (4), (5) and (6) and may be written

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_1' + \mathbf{V}_2', \tag{37}$$

where $\mathbf{V}_1'$ and $\mathbf{V}_2'$ are the interference flows created on A_1 and A_2, respectively, by the mutual cross-flow. Thus,

$$(\mathbf{V}_1' + \mathbf{V}_2 + \mathbf{V}_2') . \mathbf{n} = 0 \quad \text{on } A_1, \tag{38}$$

and

$$(\mathbf{V}_2' + \mathbf{V}_1 + \mathbf{V}_1') . \mathbf{n} = 0 \quad \text{on } A_2, \tag{39}$$

because $\mathbf{V}$ and $\mathbf{V}_1$ both satisfy (13) on A_1, and $\mathbf{V}$ and $\mathbf{V}_2$ both satisfy (13) on A_2. $\mathbf{V}_2'$ must also satisfy the Kutta condition

$$\mathbf{V}_2' . \mathbf{k} = \text{finite} \tag{40}$$

on any subsonic trailing edges of A_2. The components of the perturbation velocity associated with the fusiform body are dissimilar in magnitude near its surface and the required quadratic expression (17) for the pressure must be applied to the complete field (37), in general, because of quadratic coupling. On A_2, however, the components of $\mathbf{V}_1$ are all of order $t_1^2 \log t_1$ like the components of $\mathbf{V}_2$ and $\mathbf{V}_2'$. Equation (16) is then sufficient on A_2 or its reference surface.

The problem of the determination of $\mathbf{V}_2'$ is that of the cambered wing. The cancellation of the cross-flow of $\mathbf{V}_2 + \mathbf{V}_2'$ on A_1 with $\mathbf{V}_1'$ could be accomplished with a line of multipoles using the method of von Kármán & Moore (1932). A different approach is possible, however, which provides information required for the application of the reverse-flow relation. Lighthill (1948; see also Ward 1955*a*) has given the solution for the flow past a yawed axisymmetric fusiform body which has a ducted nose of finite diameter and a finite number of discontinuities in slope, but which otherwise fulfils the requirements of slenderness. The discontinuities in u at the surface due to the discontinuous cross-flow at the nose and to the discontinuous slopes are two-dimensional; the associated flow fields subside in a few diameters. It is found that the circumferential flow along the sections by planes $x = \text{const.}$ may be computed with unmodified slender body theory for the purpose of computing the drag from (17). Lighthill (1954) has suggested that a more general fusiform body may be built up from this solution for a finite number of jumps in slope and camber on a slender body by passing to the limit of an infinite number of smoothed discontinuities. The solution for $\mathbf{V}_1$ can be constructed in this way. The same approach would also apply for the determination of the camber-like interference flow $\mathbf{V}_1'$ produced by the wing cross-flow, which will, in general, exhibit discontinuities and vary too rapidly with x to excite an interference potential satisfying the Laplace equation in the cross-planes. Since the cross-flow is of order $t_1^2 \log t_1$, it is seen from Lighthill's solution that u_1' will be of order $t_1^2 \log t_1$ at discontinuities in cross-flow and $t_1^3 \log t_1$ when the cross-flow varies slowly with x; Lighthill's generalization would then give actual magnitudes of u_1' within this range. Further, the gross order of magnitude of the circumferential flow associated with $\mathbf{V}_1'$ on A_1 is then $t_1^2 \log t_1$ from slender body theory. The gross circumferential flow associated with $\mathbf{V}_1$ must be of order t_1 and, according to these results, may be computed from slender body theory even if the body is not slender. This information is required to discuss the accretive terms in (35).

Remarks concerning the phenomena on the surface A_2 in figure 1 (*a*) apply to both surfaces in figure 1 (*b*). Remarks concerning phenomena on A_1 apply to both surfaces in figure 1 (*c*).

Reversal of flow direction with unchanged geometry

Suppose first that the direction of flow is reversed without changing the geometry or orientation, measured with respect to the invariant co-ordinate system, of each isolated body in the arrangements considered. Interference flows are used in both directions, so that no geometric change due to interference accompanies the reversal. The required singularity distributions are, however, quite different.

Choose $T = T_1 + T_2$, $A = A_1 + A_2$, substitute (34) into (33), and use (35) and (7) in the result, to obtain

$$\oint_{A_1+A_2} [(-p_F/\rho_0)\,\mathbf{W}_R.\mathbf{n} + (p_R/\rho_0)\,\mathbf{W}_F.\mathbf{n} - \tfrac{1}{2}(\mathbf{V}_F+\mathbf{V}_R).(\mathbf{W}_F\mathbf{W}_R.\mathbf{n} - \mathbf{W}_R\mathbf{W}_F.\mathbf{n}) + \tfrac{1}{2}\mathbf{W}_F.\mathbf{V}_R(\mathbf{W}_F+\mathbf{W}_R).\mathbf{n}]\,dS = 0. \quad (41)$$

In the reverse flow

$$\mathbf{W}_R.\mathbf{n} = -\mathbf{U}_R.\mathbf{n} \quad \text{on } A_1 \text{ and } A_2, \quad (42)$$

and from (31)

$$(\mathbf{W}_F+\mathbf{W}_R).\mathbf{n} = 0. \quad (43)$$

Equation (41) becomes

$$\oint_{A_1+A_2} [(p_F/\rho_0)\,\mathbf{U}_R.\mathbf{n} - (p_R/\rho_0)\,\mathbf{U}_F.\mathbf{n} + \tfrac{1}{2}(\mathbf{V}_F+\mathbf{V}_R).(\mathbf{W}_F+\mathbf{W}_R)\,\mathbf{W}_F.\mathbf{n}]\,dS = 0. \quad (44)$$

Introduce on the surface A_1, say, of a fusiform body an orthogonal curvilinear co-ordinate system. At each point on the section of A_1 by planes $x = \text{const.}$, let $\mathbf{i}$ be the unit vector previously defined, $\mathbf{N}$ be the unit outward normal, and $\boldsymbol{\sigma}$ be the unit tangent. The scalar product in the last term of (44) may be written

$$(\mathbf{V}_F+\mathbf{V}_R).(\mathbf{W}_F+\mathbf{W}_R) = -B^2[(\mathbf{V}_F+\mathbf{V}_R).\mathbf{i}]^2 + [(\mathbf{V}_F+\mathbf{V}_R).\mathbf{N}]^2 + [(\mathbf{V}_F+\mathbf{V}_R).\boldsymbol{\sigma}]^2. \quad (45)$$

The first term is of higher order as for the pressure relation (7). In the boundary condition (13), $\mathbf{V}.\mathbf{n}$ may be replaced by $\mathbf{V}.\mathbf{N}$ in linear theory, so that

$$(\mathbf{V}_F+\mathbf{V}_R).\mathbf{N} = 0. \quad (46)$$

With regard to the last term, expand $\mathbf{V}_F$ and $\mathbf{V}_R$ according to (37). For isolated closed slender bodies, the forward and reverse flows differ only in sign. Thus, even for arbitrary fusiform bodies, the circumferential components on A_1 of order t_1 vanish:

$$(\mathbf{V}_{1F}+\mathbf{V}_{1R}).\boldsymbol{\sigma} = 0. \quad (47)$$

The remaining circumferential components are of order $t_1^2 \log t_1$ so that their squares are negligible in (44). Note that no mixed products of order $t_1^3 \log t_1$ appear; terms of this order could not be discarded.

On wing surfaces the last term in (44), as well as the quadratic terms in the pressure relation, are negligible *a priori*. Equation (44) therefore reduces to

$$\oint_{A_1+A_2} p_F \mathbf{n}.\mathbf{i}\, dS = -\oint_{A_1+A_2} p_R \mathbf{n}.\mathbf{i}\, dS, \tag{48}$$

or† $$D_F = D_R \tag{49}$$

for the three arrangements.

Evidently, the drag is also invariant under flow reversal with unchanged geometry for the special cases of an isolated, arbitrary, closed fusiform body and an isolated wing. If the former is a slender body, then this result follows independently from (30). For the latter and for other quasi-cylinders as in figure 1 (*b*), the same result follows from the Ursell–Ward theorem, to which bodies it is applicable. The surfaces A may then be taken as both sides of the reference surfaces, the spatial distributions of singularities becoming surface distributions.

Reversal of flow direction with mutual distortion

Suppose next that the direction of flow is reversed without changing the geometry or orientation of each isolated body which appears in the arrangements considered. Now, however, interference flows are used only in the forward stream. In the reverse stream each body induces a cross-flow over the surface of the other, resulting in mutual geometric distortion when the bodies interfere. The reverse flow is completely defined by the geometry of the isolated bodies. The reverse-flow theorem is used to relate the drag in the forward flow of interest to the reverse flow discussed. For reasons already mentioned, it is sufficient to consider the boundary conditions and surface pressures in the reverse flow on the surfaces of the isolated bodies A_1 and A_2 instead of on the true distorted surfaces; use may then be made of (34).

Consider specifically the arrangement in figure 1 (*a*) and choose again $T = T_1 + T_2$, $A = A_1 + A_2$. If (34) is substituted into (33) and use is made of (35), the pressure relations, and the expressions

$$\mathbf{V}_R = \mathbf{V}_{1R} + \mathbf{V}_{2R}, \tag{50}$$

$$(\mathbf{W}_F + \mathbf{W}_{1R}).\mathbf{n} = 0 \quad \text{on } A_1, \tag{51}$$

there results after several cancellations

$$\oint_{A_2} [-(p_F/\rho_0)\mathbf{W}_{2R}.\mathbf{n} + (p_R/\rho_0)\mathbf{W}_F.\mathbf{n} - (p_F/\rho_0)\mathbf{W}_{1R}.\mathbf{n}]\, dS$$
$$+ \oint_{A_1} [-(p_F/\rho_0)\mathbf{W}_{1R}.\mathbf{n} + (p_R/\rho_0)\mathbf{W}_F.\mathbf{n} + \mathbf{U}_F.\mathbf{V}_F\mathbf{W}_{2R}.\mathbf{n}$$
$$+ \tfrac{1}{2}(\mathbf{V}_F + \mathbf{V}_{1R} + \mathbf{V}_{2R}).(\mathbf{W}_F + \mathbf{W}_{1R} + \mathbf{W}_{2R})\mathbf{W}_F.\mathbf{n}]\, dS = 0. \tag{52}$$

† This result must be qualified when wing edge forces are present. With use of tubular surfaces enclosing the singular edges, the result is found to hold for the drag including the contribution thereto of edge forces due to parabolic bluntness but, when the Kutta condition is satisfied in both directions, excluding the contribution thereto of edge forces due to suction.

As above, the scalar product in the last term in (52) may be written

$$(\mathbf{V}_F+\mathbf{V}_{1R}+\mathbf{V}_{2R}).(\mathbf{W}_F+\mathbf{W}_{1R}+\mathbf{W}_{2R}) = -B^2[(\mathbf{V}_F+\mathbf{V}_{1R}+\mathbf{V}_{2R}).\mathbf{i}]^2 + [(\mathbf{V}_F+\mathbf{V}_{1R}+\mathbf{V}_{2R}).\mathbf{N}]^2+[(\mathbf{V}_F+\mathbf{V}_{1R}+\mathbf{V}_{2R}).\boldsymbol{\sigma}]^2. \quad (53)$$

The first term is negligible. In the second term, $(\mathbf{V}_F+\mathbf{V}_{1R}).\mathbf{N} = 0$ and $\mathbf{V}_{2R}.\mathbf{N}$ makes a negligible contribution. In the third term, $\mathbf{V}_F$ is expanded using (37); $(\mathbf{V}_{1F}+\mathbf{V}_{1R}).\boldsymbol{\sigma} = 0$ and the remaining four components are negligible.

In the remainder of (52), use is next made of the boundary conditions satisfied by the several perturbation velocities, and there is obtained†

$$D_F \equiv -\oint_{A_1} p_F\mathbf{n}.\mathbf{i}dS-\oint_{A_2} p_F\mathbf{n}.\mathbf{i}dS = \oint_{A_1} p_R\mathbf{n}.\mathbf{i}dS + \oint_{A_2} p_R\mathbf{n}.\mathbf{i}dS-\oint_{A_1}(-\rho_0\mathbf{U}_F.\mathbf{V}_F)\frac{1}{U_R}\mathbf{V}_{2R}.\mathbf{N}dS-\oint_{A_2} p_F\frac{1}{U_R}\mathbf{V}_{1R}.\mathbf{N}dS. \quad (54)$$

Here $\mathbf{n}$ is the outward unit normal to the undistorted surfaces in forward flow and the pressure p in each integral is as (16) or (17), according to the surface. The drag in forward flow is given by (54) as the drag in reverse flow due to the reverse pressures supported on the undistorted boundaries A_1 and A_2, minus

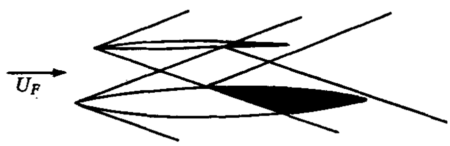

FIGURE 2. Example in which neither interference flow influences other body.

the drag in forward flow due to the forward pressures supported on the inclined stream surfaces which are induced on each boundary by the cross-flow from the other in reverse flow. Note, however, that the linear pressure relation is to be used in the latter over the surface A_1, instead of the quadratic relation generally appropriate to the nonquasi-cylindrical surface.

Suppose that the interference flow created on each surface in the forward direction does not influence the other surface, as illustrated in figure 2. An example which is appropriate to figure 1 (*b*) of this type of interference is the two-dimensional Busemann biplane or a three-dimensional counterpart. The portion of the surface of each body in the domain of dependence of the other is then not influenced by the interference. Since the domain of dependence of a body in forward flow is its domain of influence in reverse flow, the surface pressures in the last two integrals of (54) depend on only the respective isolated flows over those portions of the surfaces where the reverse cross-flow is different from zero. The drag of the arrangement in forward flow can then be computed from (54) without determining the two interference flow fields. What is required is the flow field of each isolated body in reverse flow and the surface pressure on each isolated forebody in forward flow. When the reflexions are multiple, (54) requires

† The footnote to (49) applies as well to (54). The integrals must be interpreted to include the contribution of any edge parabolic bluntness.

that the interference field be determined only over the portion of each body which appears in the domain of dependence of the other.

Equation (54) and its implications apply as well to the arrangements in figure 1 (*b*) and (*c*). In the latter case, however, the linear pressure relation is to be used in the last integral.

Subject to the edge-force qualification, (49) can be used together with (54) so that the mutual distortion occurs in the flow direction of interest. For example, Friedman & Cohen (1954) have considered two unyawed Sears–Haack bodies of revolution interfering as shown in figure 1 (*c*) in a supersonic flow (regarded here as the reverse flow). They neglected the pressures of the interference flows and took for D_R the first two integrals in the right member of (54). The remaining two integrals can be used to estimate the error. For single reflexions, the surface pressure on each body in forward flow is symmetric about its horizontal plane of symmetry and the normal cross-flow is almost antisymmetric. Therefore, the last two integrals are very small. Expansion of the cross-flow about the body axis and account for the asymmetric surface of integration gives the order of the last two integrals as no larger than $t_1^6 \log t_1$, which is negligible next to the order $t_1^4 \log t_1$ of the leading two.

Use of both volume and surface integrals

The applications of the reverse-flow relation given thus far utilize surface integrals. When, for example, only part of the complete perturbation field is to be considered in one or both flow directions, as on surfaces where superposition is permissible, it is convenient to employ volume integrations for any other bodies which are not quasi-cylinders, in order to use the drag decomposition discussed. Two such applications will now be presented in connexion with the arrangement in figure 1 (*a*).

Choose $T = T_1 + T_2$ in (33) and use (34) in the second volume integral. After comparison of the first with (32), there is obtained

$$\oint_{A_2} (\mathbf{U}_F.\mathbf{V}_F\mathbf{W}_R.\mathbf{n} - \mathbf{U}_R.\mathbf{V}_R\mathbf{W}_F.\mathbf{n})\,dS + U_F \int_{T_1} (\mathbf{i}\times\boldsymbol{\omega}_R.\mathbf{V}_F + \mathbf{i}\times\boldsymbol{\omega}_F.\mathbf{V}_R + f_R\mathbf{i}.\mathbf{V}_F + f_F\mathbf{i}.\mathbf{V}_R)\,dT = 0. \quad (55)$$

The first application is concerned with a rearrangement of (54). Let

$$\mathbf{V}_F = \mathbf{V}_{1F} + \mathbf{V}'_{1F} + \mathbf{V}_{2F} + \mathbf{V}'_{2F} \quad \text{and} \quad \mathbf{V}_R = \mathbf{V}_{1R}. \quad (56)$$

Equations (56) are substituted in (55) and use is made of the facts that $\boldsymbol{\omega}_{2F}$, $\boldsymbol{\omega}'_{2F}$, $_{2F}$ and f'_{2F} vanish over T_1, and that the interference effects among the singularities distributed over T_1 vanish separately in virtue of (32). Insertion of (16) and (31) in the result gives

$$\oint_{A_2} p_{1R}\mathbf{n}.\mathbf{i}\,dS - \oint_{A_2} p_F \frac{1}{U_R}\mathbf{V}_{1R}.\mathbf{N}\,dS = \rho_0 \int_{T_1} (\mathbf{i}\times\boldsymbol{\omega}_{1R} + f_{1R}\mathbf{i}).(\mathbf{V}_{2F} + \mathbf{V}'_{2F})\,dT. \quad (57)$$

Comparison with the right member of (54) shows that (57) re-expresses the sum of the fourth integral plus the contribution of p_{1R} to the second integral as a drag due to the cross-flow and pressure of $\mathbf{V}_{2F}$ and $\mathbf{V}'_{2F}$ on the known distribution of singularities representing the isolated body in reverse flow. The remaining part of the second integral in the right member of (54) is the drag of the isolated wing in reverse flow. When (57) is used in (54), the flow field of the fusiform body in reverse flow is required only over its surface. But the complete flow field of the isolated wing in both forward and reverse flow is then needed. As has already been discussed, $\mathbf{V}'_{2F}$ vanishes over T_1 for single interference reflexions. The two surface integrals over A_1 in the right member of (54) can also be written as volume integrals over T_1 with the aid of (34).

The second application is concerned with the lift of the arrangement in figure 1 (*a*). The lift in forward flow is that supported by the wing and follows by superposing the pressure due to $\mathbf{V}_{2F}$ and $\mathbf{V}'_{2F}$. The latter interference lift, given by

$$L'_F = 2\int_{\mathscr{A}_2} p'_{2F}\,dS, \tag{58}$$

where $\mathscr{A}_2$ is the lower side of A_2, can be obtained with a technique suggested by one used for isolated wings. Let $\mathbf{V}^*_{2R}$ be the perturbation velocity induced by a flat plate in reverse flow which has an angle of attack α^* and the same plan-form as the actual wing. To obtain this lift, the Kutta condition must be fulfilled by $\mathbf{V}^*_{2R}$ on any subsonic reverse trailing edges to exclude edge force contributions. If the actual wing is without camber and twist, $\mathbf{V}^*_{2R}$ can correspond to the odd part of the actual reverse-flow potential of the isolated wing, and α^* can be the actual angle of attack. Let

$$\mathbf{V}_F = \mathbf{V}_{1F}+\mathbf{V}'_{1F}+\mathbf{V}'_{2F}, \quad \mathbf{V}_R = \mathbf{V}^*_{2R}, \tag{59}$$

and substitute in (55). The vanishing of some vorticity and source terms over T_1 is then noted, the boundary conditions

$$\mathbf{W}_F.\mathbf{n} = 0 \quad \text{on } A_2, \tag{60}$$

and

$$\mathbf{W}_R.\mathbf{n} = \begin{cases} U_R\alpha^* & \text{on lower } A_2, \\ -U_R\alpha^* & \text{on upper } A_2, \end{cases} \tag{61}$$

are utilized, and (16) is inserted. The contributions of p_{1F} and p'_{1F} vanish in the leading term because of (61), p'_{2F} is odd in z, and α^* is constant. The result is seen to be

$$\alpha^*L'_F \equiv 2\alpha^*\int_{\mathscr{A}_2} p'_{2F}\,dS = -\rho_0\int_{T_1}[\mathbf{i}\times(\boldsymbol{\omega}_{1F}+\boldsymbol{\omega}'_{1F}).\,V^*_{2R}+(f_{1F}+f'_{1F})\,\mathbf{i}.\mathbf{V}^*_{2R}]\,dT. \tag{62}$$

The interference lift in forward flow is expressed here in terms of an interference drag supported by the forward singularities interior to the interfering body, and due to the cross-flow and streamwise perturbations of the plate wing in reverse flow. For single reflexions, $\boldsymbol{\omega}'_{1F}$ and f'_{1F} vanish in the right member and the interference lift is then determined without knowledge of the interference flows, as before. In fact, the lift can still be determined in this way if $\mathbf{V}'_2$ does influence the fusiform body.

The identity (34) does not transform the volume integrals in the right members of (57) and (62) to surface integrals involving only the mixed pressure-slope products suggested by the respective integrands unless the cubic terms are suppressed, as for quasi-cylinders; the accretive terms could be evaluated, but, for them to vanish, it is necessary that the effects of the isolated body appear in the final expressions for both flow directions (cf. (45) and (53)). This consequence of the facts that the body pressure relation is non-linear and variations across the body cannot be neglected is otherwise simply illustrated in (57), say, by setting $\boldsymbol{\omega}_{1R} = 0$ and $f_{1R}dT = U_R S'(x)\,dx$ for a line of sources representing a slender body; there results a line integral of the wing-induced linear pressure and body area along its axis. For applications such as the two just given, however, transformation is neither necessary, since boundary conditions are not in question, nor desirable, since the evaluation of surface integrals is generally more troublesome than the evaluation of the line integrals which commonly appear in specific problems.

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